

# RATIONALITY OF FIELDS OF INVARIANTS FOR SOME REPRESENTATIONS OF $\mathrm{SL}_2 \times \mathrm{SL}_2$

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**ABSTRACT.** We prove that the quotient by  $\mathrm{SL}_2 \times \mathrm{SL}_2$  of the space of bidegree  $(a, b)$  curves on  $\mathbb{P}^1 \times \mathbb{P}^1$  is rational when  $ab$  is even and  $a \neq b$ .

## 1. INTRODUCTION

The main objective of this article is to give a simple proof that the fields of invariants are rational for some irreducible representations of  $\mathrm{SL}_2 \times \mathrm{SL}_2$ . Such representations are realized as the spaces  $V_{a,b} = H^0(\mathcal{O}_Q(a, b))$  of bi-forms of bidegree  $(a, b)$  on the surface  $Q = \mathbb{P}^1 \times \mathbb{P}^1$ . By symmetry we may restrict to the range  $a \leq b$ . In [6] Shepherd-Barron proved that  $\mathbb{P}V_{3,b}/\mathrm{SL}_2 \times \mathrm{SL}_2$  with  $b$  even is rational by analyzing transvectants for bi-forms. The case  $a = 1, b$  even  $\geq 10$  is also settled by him in another paper [7]. We shall prove the following.

**Theorem 1.1.** *The quotient  $|O_Q(a, b)|/\mathrm{SL}_2 \times \mathrm{SL}_2$  is rational when  $a < b$  and  $ab$  is even.*

Let  $V_d$  denote the  $\mathrm{SL}_2$ -representation  $H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ . For most  $(a, b)$  our proof is based on the following simple idea: we identify  $V_{a,b}$  with  $V_a \otimes V_b = \mathrm{Hom}(V_a^\vee, V_b)$ , and consider the natural fibration

$$(1.1) \quad \mathrm{Hom}(V_a^\vee, V_b) \dashrightarrow \mathbb{G}(a, \mathbb{P}V_b)$$

associating the images of linear maps, where  $\mathbb{G}(a, \mathbb{P}V_b)$  is the Grassmannian of  $a$ -planes in  $\mathbb{P}V_b$ . This is birationally a vector bundle on which the first factor of  $\mathrm{SL}_2 \times \mathrm{SL}_2$  acts fiberwisely and the second factor acts equivariantly. Starting from (1.1), we compare several fibrations, and finally reduce the problem to the rationality of  $\mathbb{P}V_b/\mathrm{SL}_2$  due to Katsylo and Bogomolov [3], [4], [1].

Although we have the fibration (1.1) for any  $a \leq b$ , there arises difficulty in analyzing it in the following cases:

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- When  $ab$  is odd, a Brauer-Severi scheme over  $\mathbb{G}(a, \mathbb{P}V_b)/\mathrm{PGL}_2$  becomes birationally nontrivial;
- When  $a = b$ ,  $\mathbb{G}(a, \mathbb{P}V_b)$  is one point;
- When  $a = 1$ , we have no fibration over  $\mathbb{G}(a, \mathbb{P}V_b)$ ;
- For few other  $(a, b)$ ,  $\mathrm{PGL}_2$  does not act almost freely on some of relevant spaces.

The first two cases, excluded from Theorem 1.1, are subject of future study. For the third case (with  $b$  even) we just add few supplements to the result of [7], mainly using transvectants. To study the last case, we identify  $\mathbb{P}V_{a,b}$  birationally with the space of parametrized rational curves of degree  $b$  in  $\mathbb{P}^a$ . We have actually  $a = 2$  in the relevant cases, and the rationality is proved by studying rational plane cubics and quartics.

To conclude the introduction, we note that the above idea would apply more generally to some irreducible representations of product groups. It relies primarily on that we have sufficient stable rationality results for both factors of the representation.

Throughout this article we work over the complex numbers. In §2 we prove Theorem 1.1 for  $a > 1$ ,  $b > 4$ . In §3 and §4 we treat the remaining few cases in ad hoc ways to simplify the statement of Theorem 1.1.

## 2. FIBRATION OVER GRASSMANNIAN

Let us begin with a general setting. Let  $V, W$  be representations of algebraic groups  $G, H$  respectively. We assume that  $d+1 = \dim V \leq \dim W$ . The tensor product  $V \otimes W$  is a representation of  $G \times H$ . By identifying  $V \otimes W$  with  $\mathrm{Hom}(V^\vee, W)$  and considering the images of linear maps  $V^\vee \rightarrow W$ , we obtain a fibration

$$(2.1) \quad V \otimes W \dashrightarrow \mathbb{G}(d, \mathbb{P}W)$$

over the Grassmannian. If we denote by  $\mathcal{E} \rightarrow \mathbb{G}(d, \mathbb{P}W)$  the universal subbundle of rank  $d+1$ , then by (2.1)  $V \otimes W$  becomes  $G \times H$ -equivariantly birational to the bundle  $V \otimes \mathcal{E}$ . Here  $G$  acts linearly on  $V$  and  $H$  acts equivariantly on the bundle  $\mathcal{E}$ . In particular, we have

$$(2.2) \quad \mathbb{P}(V \otimes W)/G \times H \sim \mathbb{P}(V \otimes \mathcal{E})/G \times H.$$

(When  $\dim V \geq \dim W$ , we instead associate the kernels of linear maps  $V^\vee \rightarrow W$  to obtain a similar correspondence for the universal quotient bundle over a Grassmannian.)

To attack the rationality problem for  $\mathbb{P}(V \otimes W)/G \times H$  via (2.2), one first tries to apply the no-name lemma to the  $H$ -linearized bundle  $\mathcal{E}$  (replacing  $H$  with its quotient). If this was successful, we would have

$$(2.3) \quad \mathbb{P}(V \otimes W)/G \times H \sim (\mathbb{G}(d, \mathbb{P}W)/H) \times (\mathbb{P}(V \otimes \mathbb{C}^{d+1})/G),$$

where  $G$  acts on  $\mathbb{C}^{d+1}$  trivially. Thus the problem could be reduced to the rationality of  $\mathbb{P}(V^{\oplus d+1})/G$  and a stable rationality of  $\mathbb{G}(d, \mathbb{P}W)/H$ . The latter might be in turn deduced from a stable rationality of  $\mathbb{P}W/H$ , using the no-name lemma for  $\mathcal{E}$  and considering the natural Grassmannian fibration  $\mathbb{P}\mathcal{E} \rightarrow \mathbb{P}W$ .

In the rest of this section we shall prove Theorem 1.1 for  $a > 1, b > 4$  by carrying out this process for  $G = H = \mathrm{SL}_2$ ,  $V = V_a$ , and  $W = V_b$ . We first study the case  $b$  even in §2.1. The case  $(a, b) = (\text{even}, \text{odd})$  may be treated similarly with few minor modifications (§2.2). Before proceeding, we note here that

**Lemma 2.1.** *The group  $\mathrm{PGL}_2$  acts almost freely on  $\mathbb{G}(a, \mathbb{P}V_b)$  when  $b > 4$ .*

*Proof.* First observe that for a general point  $p \in \mathbb{P}V_b$  the orbit  $\mathrm{PGL}_2 \cdot p$  does not intersect outside  $p$  with general codimension  $\geq 4$  planes through  $p$ . This can be seen, e.g., by considering the projection  $\mathbb{P}V_b \dashrightarrow \mathbb{P}^{b-1}$  from  $p$ . Since  $p$  has no nontrivial stabilizer in  $\mathrm{PGL}_2$ , from this the lemma follows in the case  $b - a \geq 4$ . By the dualities

$$\mathbb{G}(a, \mathbb{P}V_b) \simeq \mathbb{G}(a, \mathbb{P}V_b^\vee) \simeq \mathbb{G}(b - a - 1, \mathbb{P}V_b),$$

the range  $a \geq 3$  is also covered. Except for  $(a, b) = (2, 5)$  either  $a \geq 3$  or  $b - a \geq 4$  holds. The left case  $(a, b) = (2, 5)$  may be reduced to the fact that a general rational plane quintic has no nontrivial stabilizer in  $\mathrm{PGL}_3$ .  $\square$

**2.1. The case  $b$  even.** We assume that  $1 < a < b$  and  $b$  is even with  $b \geq 6$ . Then  $-1 \in \mathrm{SL}_2$  acts trivially on  $V_b$  so that  $V_b$  is a  $\mathrm{PGL}_2$ -representation. In particular, the bundle  $\mathcal{E}$  on  $\mathbb{G}(a, \mathbb{P}V_b)$  is  $\mathrm{PGL}_2$ -linearized. By Lemma 2.1 we may apply the no-name lemma to  $\mathcal{E}$ . Therefore we have

$$(2.4) \quad \mathbb{P}(V_a \otimes \mathcal{E})/\mathrm{PGL}_2 \times \mathrm{PGL}_2 \sim (\mathbb{G}(a, \mathbb{P}V_b)/\mathrm{PGL}_2) \times (\mathbb{P}(V_a \otimes \mathbb{C}^{a+1})/\mathrm{PGL}_2).$$

The quotient  $\mathbb{P}(V_a \otimes \mathbb{C}^{a+1})/\mathrm{PGL}_2$  is rational: for example, it is birationally the space of rational normal curves in  $\mathbb{P}V_a^\vee$ , which is rational by Hirschowitz [2]. Therefore we have

$$(2.5) \quad \mathbb{P}V_{a,b}/\mathrm{PGL}_2 \times \mathrm{PGL}_2 \sim (\mathbb{G}(a, \mathbb{P}V_b)/\mathrm{PGL}_2) \times \mathbb{P}^{(a+1)^2-4}.$$

We shall show that  $\mathbb{G}(a, \mathbb{P}V_b)/\mathrm{PGL}_2$  is stably rational of small level. By the no-name lemma for  $\mathcal{E}$  again we have

$$(2.6) \quad \mathbb{P}\mathcal{E}/\mathrm{PGL}_2 \sim (\mathbb{G}(a, \mathbb{P}V_b)/\mathrm{PGL}_2) \times \mathbb{P}^a.$$

On the other hand, consider the projection

$$(2.7) \quad \pi: \mathbb{P}\mathcal{E} \rightarrow \mathbb{P}V_b, \quad (P, x) \mapsto x,$$

where  $P$  is an  $a$ -plane in  $\mathbb{P}V_b$  and  $x$  is a point of  $P$ . The  $\pi$ -fiber over an  $x \in \mathbb{P}V_b$  is the sub Grassmannian in  $\mathbb{G}(a, \mathbb{P}V_b)$  consisting of  $a$ -planes containing  $x$ . Therefore, if  $\mathcal{G} \rightarrow \mathbb{P}V_b$  is the universal quotient bundle of rank  $b$ ,  $\pi$  is

canonically isomorphic to the relative Grassmannian  $\mathbb{G}(a-1, \mathbb{P}\mathcal{G})$  for  $\mathcal{G}$ . Since  $\mathcal{G}$  is  $\mathrm{PGL}_2$ -linearized and  $\mathrm{PGL}_2$  acts almost freely on  $\mathbb{P}V_b$ , we may apply the no-name lemma to  $\mathcal{G}$  to see that  $\mathbb{G}(a-1, \mathbb{P}\mathcal{G})$  is  $\mathrm{PGL}_2$ -birational to  $\mathbb{G}(a-1, \mathbb{P}^{b-1}) \times \mathbb{P}V_b$ , where  $\mathrm{PGL}_2$  acts on  $\mathbb{G}(a-1, \mathbb{P}^{b-1})$  trivially. In particular, we have

$$(2.8) \quad \mathbb{P}\mathcal{E}/\mathrm{PGL}_2 \sim \mathbb{G}(a-1, \mathbb{P}^{b-1}) \times (\mathbb{P}V_b/\mathrm{PGL}_2) \sim \mathbb{P}^{a(b-a)} \times (\mathbb{P}V_b/\mathrm{PGL}_2).$$

Combining (2.5), (2.6), and (2.8), and noticing that  $(a+1)^2 - 4 > a$  for  $a > 1$ , we conclude that

$$(2.9) \quad \mathbb{P}V_{a,b}/\mathrm{PGL}_2 \times \mathrm{PGL}_2 \sim \mathbb{P}^N \times (\mathbb{P}V_b/\mathrm{PGL}_2)$$

with  $N = (a+1)^2 - 4 - a + a(b-a) > 0$ . The quotient  $\mathbb{P}V_b/\mathrm{PGL}_2$  is rational by Katsylo and Bogomolov [4], [1]. Therefore  $\mathbb{P}V_{a,b}/\mathrm{SL}_2 \times \mathrm{SL}_2$  is rational for  $1 < a < b$  with  $b \geq 6$  even.

**2.2. The case  $a$  even and  $b$  odd.** Let  $1 < a < b$  with  $a$  even and  $b \geq 5$  odd. In the present case  $-1 \in \mathrm{SL}_2$  acts on  $V_b$  by  $-1$ , and hence  $\mathcal{E}$  is not  $\mathrm{PGL}_2$ -linearized. Nevertheless we may twist  $\mathcal{E}$  by the line bundle  $\mathcal{L} = \det \mathcal{E}$ . Since  $\mathcal{E}$  has odd rank  $a+1$ ,  $-1 \in \mathrm{SL}_2$  acts on  $\mathcal{L}$  by  $-1$ . Therefore  $\mathcal{E}' = \mathcal{E} \otimes \mathcal{L}$  is a  $\mathrm{PGL}_2$ -linearized bundle. Since  $\mathbb{P}(V_a \otimes \mathcal{E})$  is canonically isomorphic to  $\mathbb{P}(V_a \otimes \mathcal{E}')$ , replacing  $\mathcal{E}$  by  $\mathcal{E}'$  in the argument we obtain the same equivalence as (2.4). Then (2.5) follows similarly. Replacing  $\mathcal{E}$  with  $\mathcal{E}'$ , we also have (2.6). Finally, let  $\mathcal{O}_{\mathbb{P}V_b}(1)$  be the hyperplane bundle on  $\mathbb{P}V_b$ . The element  $-1 \in \mathrm{SL}_2$  acts on  $\mathcal{O}_{\mathbb{P}V_b}(1)$  by  $-1$ , so that the bundle  $\mathcal{G}' = \mathcal{G} \otimes \mathcal{O}_{\mathbb{P}V_b}(1)$  is  $\mathrm{PGL}_2$ -linearized. Replacing  $\mathcal{G}$  by  $\mathcal{G}'$ , we obtain (2.8) and (2.9) as well. Since  $\mathbb{P}V_b/\mathrm{PGL}_2$  for odd  $b$  is rational by Katsylo [3], Theorem 1.1 is proved in the present case.

### 3. RATIONAL SPACE CURVES

In the rest of this article we study the cases excluded from the previous §2 to complete the proof of Theorem 1.1. The cases  $(a, b) = (3, 4)$  and  $a = 1$ ,  $b = 2n \geq 10$  are settled by Shepherd-Barron in [6] and [7] respectively. (In [7] he proved the rationality of  $\mathbb{G}(1, \mathbb{P}V_b)/\mathrm{SL}_2$ , which by either (2.2) or (3.3) is birational to  $\mathbb{P}V_{1,b}/\mathrm{SL}_2 \times \mathrm{SL}_2$ .) Hence the cases to be considered are

$$(a, b) = (2, 3), (2, 4), (1, 4), (1, 6), (1, 8).$$

In this section we study the first three cases by geometric approaches. In §3.1 we identify  $|O_Q(a, b)|$  birationally with the space of some parametrized rational space curves for any  $(a, b)$ . Using that description, we study the cases  $(a, b) = (2, 3)$  and  $(2, 4)$  in §3.2 and §3.3 respectively. The case  $(a, b) = (1, 4)$  is treated independently in §3.4.

**3.1. Rational space curves.** Let  $a, b > 0$  be any positive integers. To a general curve  $C$  on  $Q = \mathbb{P}^1 \times \mathbb{P}^1$  of bidegree  $(a, b)$  we may associate a morphism  $\phi_C: \mathbb{P}^1 \rightarrow \mathbb{P}V_b = |\mathcal{O}_{\mathbb{P}^1}(b)|$  by regarding  $C$  as a family of  $b$  points on the second factor  $\mathbb{P}^1$  parametrized by the first factor  $\mathbb{P}^1$ .

**Lemma 3.1.** *The curve  $\phi_C(\mathbb{P}^1)$  has degree  $a$ , i.e.,  $\phi_C^* \mathcal{O}_{\mathbb{P}V_b}(1) \simeq \mathcal{O}_{\mathbb{P}^1}(a)$ .*

*Proof.* By the Riemann-Hurwitz formula the first projection  $C \rightarrow \mathbb{P}^1$  has  $r = 2g_C - 2 + 2b$  branch points where  $g_C$  is the genus of  $C$ . Substituting  $g_C = (a-1)(b-1)$ , we have  $r = 2a(b-1)$ . These branch points on  $\mathbb{P}^1$  correspond to the intersection of  $\phi_C(\mathbb{P}^1)$  with the discriminant hypersurface  $D$  in  $\mathbb{P}V_b$ . Since  $D$  has degree  $2(b-1)$ ,  $\phi_C(\mathbb{P}^1)$  has degree  $a$ .  $\square$

Conversely, given a general morphism  $\phi: \mathbb{P}^1 \rightarrow \mathbb{P}V_b$  of degree  $a$ , we obtain a curve on  $\mathbb{P}^1 \times \mathbb{P}^1$  by pulling back the universal divisor on  $\mathbb{P}V_b \times \mathbb{P}^1$ . Reversing the above calculation, we see that  $C$  has bidegree  $(a, b)$ .

Let  $U_{a,b}$  be the space of morphisms  $\mathbb{P}^1 \rightarrow \mathbb{P}V_b$  of degree  $a$ , on which  $\mathrm{PGL}_2 \times \mathrm{PGL}_2$  acts as follows: the first factor  $\mathrm{PGL}_2$  acts on the source  $\mathbb{P}^1$  of the morphisms, and the second factor  $\mathrm{PGL}_2$  acts on the target  $\mathbb{P}V_b$  in the natural way. Then the above construction gives a  $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ -equivariant birational map

$$(3.1) \quad \mathbb{P}V_{a,b} = |\mathcal{O}_Q(a, b)| \dashrightarrow U_{a,b}.$$

We obtain in particular that

$$(3.2) \quad \mathbb{P}V_{a,b}/\mathrm{PGL}_2 \times \mathrm{PGL}_2 \sim U_{a,b}/\mathrm{PGL}_2 \times \mathrm{PGL}_2.$$

If we denote by  $R_{a,b}$  the space of rational curves of degree  $a$  in  $\mathbb{P}V_b$ , this may also be written as

$$(3.3) \quad \mathbb{P}V_{a,b}/\mathrm{PGL}_2 \times \mathrm{PGL}_2 \sim R_{a,b}/\mathrm{PGL}_2,$$

where  $\mathrm{PGL}_2$  acts on  $R_{a,b}$  by its action on  $\mathbb{P}V_b$ . Since  $\mathrm{PGL}_2$  as the subgroup of  $\mathrm{Aut}(\mathbb{P}V_b) \simeq \mathrm{PGL}_{b+1}$  is the stabilizer of a rational normal curve, we have

$$(3.4) \quad \mathbb{P}V_{a,b}/\mathrm{PGL}_2 \times \mathrm{PGL}_2 \sim (R_{a,b} \times R_{b,b})/\mathrm{PGL}_{b+1}.$$

Exchanging  $a$  and  $b$ , we also obtain

$$(3.5) \quad \mathbb{P}V_{a,b}/\mathrm{PGL}_2 \times \mathrm{PGL}_2 \sim R_{b,a}/\mathrm{PGL}_2 \sim (R_{b,a} \times R_{a,a})/\mathrm{PGL}_{a+1}.$$

*Remark 3.2.* The above (3.1) and the description  $\mathbb{P}V_{a,b} \sim \mathbb{P}(V_a \otimes \mathcal{E})$  in §2 are connected by considering the linear span of  $\phi_C(\mathbb{P}^1)$ , which is generically  $a$ -dimensional and in which  $\phi_C(\mathbb{P}^1)$  is a rational normal curve.

**3.2. The case  $(a, b) = (2, 3)$ .** By (3.5) it suffices to prove that  $R_{3,2}/\mathrm{PGL}_2$  is rational, where  $R_{3,2} \subset |\mathcal{O}_{\mathbb{P}^2}(3)|$  is the space of rational plane cubics and  $\mathrm{PGL}_2 \subset \mathrm{PGL}_3$  is the stabilizer of some reference smooth conic  $\Gamma$ . We may take the homogeneous coordinate  $[X, Y, Z]$  of  $\mathbb{P}^2$  and normalize  $\Gamma$  to be defined by  $XZ = Y^2$ .

Every rational plane cubic has a unique singularity. We apply the slice method for the nodal map

$$(3.6) \quad \kappa : R_{3,2} \rightarrow \mathbb{P}^2, \quad C \mapsto \mathrm{Sing}C,$$

which is clearly  $\mathrm{PGL}_2$ -equivariant. The group  $\mathrm{PGL}_2$  acts on  $\mathbb{P}^2 - \Gamma$  transitively, and the stabilizer  $G$  of the point  $p = [0, 1, 0]$  is isomorphic to  $(\mathbb{Z}/2\mathbb{Z}) \ltimes \mathbb{C}^\times$  where  $\mathbb{Z}/2\mathbb{Z}$  acts by  $[X, Y, Z] \mapsto [Z, Y, X]$  and  $\alpha \in \mathbb{C}^\times$  acts by  $[X, Y, Z] \mapsto [\alpha^{-1}X, Y, \alpha Z]$ . The fiber  $\kappa^{-1}(p)$  is an open set of the linear system  $\mathbb{P}V \subset |\mathcal{O}_{\mathbb{P}^2}(3)|$  of cubics singular at  $p$ . Hence we have

$$(3.7) \quad R_{3,2}/\mathrm{PGL}_2 \sim \mathbb{P}V/G.$$

The group  $G$  acts linearly on  $V$  and we have the following  $G$ -decomposition:

$$(3.8) \quad V = \langle XYZ \rangle \oplus \langle X^2Z, Z^2X \rangle \oplus \langle X^2Y, YZ^2 \rangle \oplus \langle X^3, Z^3 \rangle.$$

Let  $W = \langle X^2Z, Z^2X, X^2Y, YZ^2 \rangle$ ,  $W^\perp = \langle XYZ, X^3, Z^3 \rangle$ , and consider the projection  $\pi : \mathbb{P}V \dashrightarrow \mathbb{P}W$  from  $W^\perp$ . Then  $\pi$  is a  $G$ -linearized vector bundle. Since  $G$  acts on  $\mathbb{P}W$  almost freely, by the no-name method we have

$$(3.9) \quad \mathbb{P}V/G \sim \mathbb{C}^3 \times (\mathbb{P}W/G).$$

The quotient  $\mathbb{P}W/G$  is rational because it is 2-dimensional. This proves that  $\mathbb{P}V_{2,3}/\mathrm{PGL}_2 \times \mathrm{PGL}_2$  is rational.

**3.3. The case  $(a, b) = (2, 4)$ .** By (3.5) it is sufficient to show that  $R_{4,2}/\mathrm{PGL}_2$  is rational, where  $\mathrm{PGL}_2$  is the stabilizer in  $\mathrm{PGL}_3$  of some smooth conic. General rational plane quartics have three nodes. Let  $S^3\mathbb{P}^2$  be the third symmetric product of  $\mathbb{P}^2$ , and consider the nodal map

$$(3.10) \quad \kappa : R_{4,2} \dashrightarrow S^3\mathbb{P}^2, \quad C \mapsto \mathrm{Sing}C.$$

General  $\kappa$ -fibers are open sets of sub linear systems of  $|\mathcal{O}_{\mathbb{P}^2}(4)|$ . Since  $\mathrm{PGL}_2$  acts linearly on  $H^0(\mathcal{O}_{\mathbb{P}^2}(4))$ ,  $\kappa$  is birationally the projectivization of a  $\mathrm{PGL}_2$ -linearized vector bundle. Since  $\mathrm{PGL}_2$  acts on  $S^3\mathbb{P}^2$  almost freely, by the no-name lemma we have

$$(3.11) \quad R_{4,2}/\mathrm{PGL}_2 \sim \mathbb{P}^5 \times (S^3\mathbb{P}^2/\mathrm{PGL}_2).$$

Using the slice method (in the converse direction), we see that

$$(3.12) \quad S^3\mathbb{P}^2/\mathrm{PGL}_2 \sim (S^3\mathbb{P}^2 \times |\mathcal{O}_{\mathbb{P}^2}(2)|)/\mathrm{PGL}_3.$$

We then apply the slice method to the projection  $S^3\mathbb{P}^2 \times |\mathcal{O}_{\mathbb{P}^2}(2)| \rightarrow S^3\mathbb{P}^2$ . The group  $\mathrm{GL}_3$  acts on  $S^3\mathbb{P}^2$  almost transitively, and the stabilizer  $G$  of

$$(3.13) \quad p_1 + p_2 + p_3 = [1, 0, 0] + [0, 1, 0] + [0, 0, 1]$$

is isomorphic to  $\mathfrak{S}_3 \ltimes (\mathbb{C}^\times)^3$  where  $\mathfrak{S}_3$  acts by the permutations of  $X, Y, Z$  and  $(\mathbb{C}^\times)^3$  is the torus of diagonal matrices. Then we have

$$(3.14) \quad (S^3\mathbb{P}^2 \times |\mathcal{O}_{\mathbb{P}^2}(2)|)/\mathrm{PGL}_3 \sim |\mathcal{O}_{\mathbb{P}^2}(2)|/G \sim H^0(\mathcal{O}_{\mathbb{P}^2}(2))/G.$$

The  $G$ -representation  $H^0(\mathcal{O}_{\mathbb{P}^2}(2))$  is decomposed as

$$(3.15) \quad H^0(\mathcal{O}_{\mathbb{P}^2}(2)) = \langle X^2, Y^2, Z^2 \rangle \oplus \langle XY, YZ, ZX \rangle.$$

We set  $W = \langle X^2, Y^2, Z^2 \rangle$  and  $W^\perp = \langle XY, YZ, ZX \rangle$ . The group  $G$  acts on  $W$  almost transitively, so that we may apply the slice method to the projection  $H^0(\mathcal{O}_{\mathbb{P}^2}(2)) \rightarrow W$  from  $W^\perp$ . Hence for the stabilizer  $H \subset G$  of a general point of  $W$  we have

$$(3.16) \quad H^0(\mathcal{O}_{\mathbb{P}^2}(2))/G \sim W^\perp/H.$$

Then  $W^\perp/H$  is birational to  $\mathbb{C}^\times \times (\mathbb{P}W^\perp/H)$ , and  $\mathbb{P}W^\perp/H$  is rational because it is 2-dimensional. This completes the proof that  $\mathbb{P}V_{2,4}/\mathrm{PGL}_2 \times \mathrm{PGL}_2$  is rational.

**3.4. The case  $(a, b) = (1, 4)$ .** The quotient  $\mathbb{P}V_{1,4}/\mathrm{PGL}_2 \times \mathrm{PGL}_2$  is birational to  $\mathbb{G}(1, \mathbb{P}V_4)/\mathrm{PGL}_2$  by (3.3). Since  $V_4 \simeq V_4^\vee$  as  $\mathrm{SL}_2$ -representations, we have a  $\mathrm{PGL}_2$ -equivariant isomorphism  $\mathbb{G}(1, \mathbb{P}V_4) \simeq \mathbb{G}(1, \mathbb{P}V_4^\vee)$ . By projecting the standard rational normal curve in  $\mathbb{P}V_4^\vee$  from lines, we obtain a birational map

$$(3.17) \quad \mathbb{G}(1, \mathbb{P}V_4^\vee)/\mathrm{PGL}_2 \dashrightarrow R_{4,2}/\mathrm{PGL}_3.$$

Thus the problem is reduced to the rationality of  $R_{4,2}/\mathrm{PGL}_3$ .

We apply the slice method to the nodal map (3.10), which we now regard as a  $\mathrm{GL}_3$ -equivariant map. We reuse the notations  $p_1 + p_2 + p_3, G$  from §3.3. Then for the linear system  $\mathbb{P}V$  of quartics singular at  $p_1 + p_2 + p_3$  we have

$$(3.18) \quad R_{4,2}/\mathrm{PGL}_3 \sim \mathbb{P}V/G \sim V/G.$$

In terms of the coordinate  $[X, Y, Z]$  the  $G$ -representation  $V$  is decomposed as

$$(3.19) \quad V = \langle X^2Y^2, Y^2Z^2, Z^2X^2 \rangle \oplus \langle X^2YZ, Y^2ZX, Z^2XY \rangle.$$

The rest of the proof is similar to the final step in §3.3: we may use the slice method for the projection of  $V$  from either irreducible summand, and then resort to Castelnuovo's theorem to see that  $V/G$  is rational. Thus  $\mathbb{P}V_{1,4}/\mathrm{PGL}_2 \times \mathrm{PGL}_2$  is rational.

## 4. TRANSVECTANT

In this section we treat the cases  $(a, b) = (1, 6), (1, 8)$ . We first recall in §4.1 some basic facts about transvectants for biforms. In §4.2 and §4.3 we study those cases by applying the method of double fibration ([1]) to certain transvectants.

**4.1. Transvectants for biforms.** For two representations  $V_{a,b}, V_{a',b'}$  of  $\mathrm{SL}_2 \times \mathrm{SL}_2$ , their tensor product is

$$(4.1) \quad V_{a,b} \otimes V_{a',b'} = (V_a \boxtimes V_b) \otimes (V_{a'} \boxtimes V_{b'}) = (V_a \otimes V_{a'}) \boxtimes (V_b \otimes V_{b'}).$$

Applying the Clebsch-Gordan decomposition for  $\mathrm{SL}_2$ ,

$$(4.2) \quad V_d \otimes V_{d'} = \bigoplus_{r=0}^{d''} V_{d+d'-2r}, \quad d'' = \min\{d, d'\},$$

we obtain the irreducible decomposition

$$(4.3) \quad V_{a,b} \otimes V_{a',b'} = \bigoplus_{r,s} V_{a+a'-2r, b+b'-2s},$$

where  $0 \leq r \leq \min\{a, a'\}$  and  $0 \leq s \leq \min\{b, b'\}$ . By this decomposition we have an  $\mathrm{SL}_2 \times \mathrm{SL}_2$ -equivariant bilinear map

$$(4.4) \quad T^{(r,s)} : V_{a,b} \times V_{a',b'} \rightarrow V_{a+a'-2r, b+b'-2s},$$

unique up to scalar multiplication. Let  $T^{(r)} : V_d \times V_{d'} \rightarrow V_{d+d'-2r}$  be the  $r$ -th *transvectant*, i.e., an  $\mathrm{SL}_2$ -bilinear map associated to (4.2). Then a standard argument in linear algebra shows that  $T^{(r,s)}$  is given (up to constant) by

$$(4.5) \quad T^{(r,s)}(P_1 \otimes P_2, P'_1 \otimes P'_2) = T^{(r)}(P_1, P'_1) \otimes T^{(s)}(P_2, P'_2),$$

where  $P_1 \in V_{a,0} = V_a, P_2 \in V_{0,b} = V_b, P'_1 \in V_{a',0} = V_{a'}$ , and  $P'_2 \in V_{0,b'} = V_{b'}$ .

Let  $[X, Y]$  be the homogeneous coordinate of  $\mathbb{P}^1$ . The transvectant  $T^{(r)}$  is given explicitly by the following (cf. [5]):

$$(4.6) \quad T^{(r)}(P, P') = \sum_{i=0}^r (-1)^i \binom{r}{i} \frac{\partial^r P}{\partial X^{r-i} \partial Y^i} \frac{\partial^r P'}{\partial X^i \partial Y^{r-i}}.$$

When  $r = d' \leq d$  in particular,  $T^{(d')}(P, P')$  is called the *apolar covariant* and calculated by substituting  $-\frac{\partial}{\partial Y}, \frac{\partial}{\partial X}$  respectively into  $X, Y$  in  $P'$ , applying that differential polynomial to  $P$ , and then multiplying it by  $d'!$ .

From (4.5) and (4.6) we may calculate the  $(r, s)$ -th transvectant  $T^{(r,s)}$  explicitly in terms of the bi-homogeneous coordinate  $([X_1, Y_1], [X_2, Y_2])$  of  $\mathbb{P}^1 \times \mathbb{P}^1$ . For example, when  $a = a' = 1$  and  $b \geq b'$ , we have

$$(4.7) \quad T^{(1,s)}(X_1 \otimes P + Y_1 \otimes Q, X_1 \otimes P' + Y_1 \otimes Q') = T^{(s)}(P, Q') - T^{(s)}(Q, P'),$$

where  $s \leq b', P, Q \in V_{0,b} = V_b$ , and  $P', Q' \in V_{0,b'} = V_{b'}$ .



**4.2. The case  $(a, b) = (1, 6)$ .** We shall apply the method of double fibration ([1]) to the bi-apolar covariant

$$(4.8) \quad T^{(1,2)} : V_{1,6} \times V_{1,2} \rightarrow V_{0,4}.$$

Note that  $\dim V_{1,2} = \dim V_{0,4} + 1$ . The image of  $V_{1,6} \rightarrow \text{Hom}(V_{1,2}, V_{0,4})$  given by  $H \mapsto T^{(1,2)}(H, \bullet)$  is not contained in the degeneracy locus: for example, take  $H$  to be  $X_1 X_2^3 Y_2^3 + Y_1 (X_2^4 Y_2^2 + X_2^2 Y_2^4)$ . Thus the  $\text{PGL}_2 \times \text{PGL}_2$ -equivariant map

$$(4.9) \quad \varphi : \mathbb{P}V_{1,6} \dashrightarrow \mathbb{P}V_{1,2}, \quad \mathbb{C}H \mapsto \text{Ker}(T^{(1,2)}(H, \bullet)),$$

is well-defined. Note in passing that the  $\varphi$ -image of the above  $X_1 X_2^3 Y_2^3 + Y_1 (X_2^4 Y_2^2 + X_2^2 Y_2^4)$  defines a smooth curve on  $\mathbb{P}^1 \times \mathbb{P}^1$ .

**Lemma 4.1.** *The group  $\text{PGL}_2 \times \text{PGL}_2$  acts transitively on the open locus  $U$  in  $\mathbb{P}V_{1,2}$  of smooth curves. If we take  $C \in U$  to be  $X_1 Y_2^2 + Y_1 X_2^2 = 0$ , its stabilizer  $G$  is isomorphic to  $(\mathbb{Z}/2\mathbb{Z}) \ltimes \mathbb{C}^\times$  where  $\mathbb{Z}/2\mathbb{Z}$  acts by  $[X_i, Y_i] \mapsto [Y_i, X_i]$  and  $\alpha \in \mathbb{C}^\times$  acts by  $[X_1, Y_1] \mapsto [X_1, \alpha^2 Y_1]$ ,  $[X_2, Y_2] \mapsto [X_2, \alpha Y_2]$ .*

*Proof.* By the birational map (3.1)  $U$  is mapped isomorphically to the space of linear embeddings  $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}V_2$  such that  $\phi(\mathbb{P}^1)$  is transverse to the diagonal conic  $\Gamma \subset \mathbb{P}V_2$ . The first assertion holds because the lines in  $\mathbb{P}V_2$  transverse to  $\Gamma$  are all  $\text{PGL}_2$ -equivalent. The stabilizer in  $\text{PGL}_2 \times \text{PGL}_2$  of any  $C \in U$  is mapped injectively by the projection to the second  $\text{PGL}_2$ , and its image is the stabilizer of the pencil  $\phi_C(\mathbb{P}^1)$ . Our second assertion follows from this observation and little calculation.  $\square$

By this lemma we may apply the slice method to  $\varphi$ . The  $\varphi$ -fiber over  $\mathbb{C}(X_1 Y_2^2 + Y_1 X_2^2)$  is an open set of the projectivization of the linear space

$$(4.10) \quad V = \{H \in V_{1,6}, \quad T^{(1,2)}(H, X_1 Y_2^2 + Y_1 X_2^2) = 0\}.$$

Then we have

$$(4.11) \quad \mathbb{P}V_{1,6}/\text{PGL}_2 \times \text{PGL}_2 \sim \mathbb{P}V/G,$$

where  $G$  is as described in the above lemma. The  $G$ -action on  $\mathbb{P}V$  is induced from the linear  $G$ -action on  $V$  given by

$$(4.12) \quad \alpha \in \mathbb{C}^\times : P_1(X_1, Y_1)P_2(X_2, Y_2) \mapsto \alpha^{-4}P_1(X_1, \alpha^2 Y_1)P_2(X_2, \alpha Y_2),$$

where  $P_1 \in V_{1,0}$  and  $P_2 \in V_{0,6}$ .

We express elements of  $V_{1,6}$  as  $X_1 P + Y_1 Q$ ,  $P = \sum_{i=0}^6 \binom{6}{i} \alpha_i X_2^i Y_2^{6-i}$ , and  $Q = \sum_{i=0}^6 \binom{6}{i} \beta_i X_2^i Y_2^{6-i}$ . By direct calculation using (4.7) and (4.6), we see that  $V$  is defined by

$$(4.13) \quad \alpha_i = \beta_{i+2}, \quad 0 \leq i \leq 4.$$

Then we have the  $G$ -decomposition  $V = \oplus_{i=0}^4 W_i$ , where

$$\begin{aligned} W_0 &= \langle X_1 X_2^2 Y_2^4 + Y_1 X_2^4 Y_2^2 \rangle, \\ W_1 &= \langle 10X_1 X_2^3 Y_2^3 + 3Y_1 X_2^5 Y_2, 3X_1 X_2 Y_2^5 + 10Y_1 X_2^3 Y_2^3 \rangle, \\ W_2 &= \langle 15X_1 X_2^4 Y_2^2 + Y_1 X_2^6, X_1 Y_2^6 + 15Y_1 X_2^2 Y_2^4 \rangle, \\ W_3 &= \langle X_1 X_2^5 Y_2, Y_1 X_2 Y_2^5 \rangle, \\ W_4 &= \langle X_1 X_2^6, Y_1 Y_2^6 \rangle. \end{aligned}$$

For  $i \geq 1$  the  $i$ -th summand  $W_i$  is the induced representation of the weight  $i$  scalar representation of  $\mathbb{C}^\times$ . The group  $G$  acts almost freely on  $\mathbb{P}(W_1 \oplus W_2)$ . Therefore we may apply the no-name method to the projection  $\mathbb{P}V \dashrightarrow \mathbb{P}(W_1 \oplus W_2)$  from  $W_0 \oplus W_3 \oplus W_4$  to see that

$$(4.14) \quad \mathbb{P}V/G \sim \mathbb{C}^5 \times (\mathbb{P}(W_1 \oplus W_2)/G).$$

Then  $\mathbb{P}(W_1 \oplus W_2)/G$  is 2-dimensional and hence is rational. This finishes the proof that  $\mathbb{P}V_{1,6}/\mathrm{PGL}_2 \times \mathrm{PGL}_2$  is rational.

**4.3. The case  $(a, b) = (1, 8)$ .** We want to show that the  $(1, 2)$ -th transvectant

$$(4.15) \quad T^{(1,2)} : V_{1,8} \times V_{1,4} \rightarrow V_{0,8}$$

determines a double fibration ([1]). Note that  $\dim V_{1,4} = \dim V_{0,8} + 1$ . The non-degeneracy condition is checked, e.g., by the following.

**Lemma 4.2.** *Take  $H = X_1 X_2^2 Y_2^6 + Y_1 X_2^6 Y_2^2 \in V_{1,8}$  and  $H' = X_1 Y_2^4 + Y_1 X_2^4 \in V_{1,4}$ . Then we have  $T^{(1,2)}(H, H') = 0$ , and the linear maps  $T^{(1,2)}(H, \bullet) : V_{1,4} \rightarrow V_{0,8}$  and  $T^{(1,2)}(\bullet, H') : V_{1,8} \rightarrow V_{0,8}$  are both surjective.*

*Proof.* This is verified by straightforward (but lengthy) calculation using (4.7) and (4.6). We leave it to the reader.  $\square$

Therefore by [1] the  $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ -equivariant map

$$(4.16) \quad \mathbb{P}V_{1,8} \dashrightarrow \mathbb{P}V_{1,4}, \quad \mathbb{C}H \mapsto \mathrm{Ker}(T^{(1,2)}(H, \bullet)),$$

is well-defined, dominant, and birationally a projective space bundle. Explicitly, let

$$(4.17) \quad \mathcal{H} = \{(H, \mathbb{C}H') \in V_{1,8} \times \mathbb{P}V_{1,4}, T^{(1,2)}(H, H') = 0\}.$$

Then  $\mathcal{H}$  is generically a sub vector bundle of  $V_{1,8} \times \mathbb{P}V_{1,4}$  invariant under the  $\mathrm{SL}_2 \times \mathrm{SL}_2$ -linearization. By the lemma  $\mathcal{H}$  has the expected rank 9, and the natural projection  $\mathbb{P}\mathcal{H} \rightarrow \mathbb{P}V_{1,8}$  is birational. Since  $\mathrm{SL}_2 \times \mathrm{PGL}_2$  acts linearly on  $V_{1,8}$ ,  $\mathcal{H}$  is in fact  $\mathrm{SL}_2 \times \mathrm{PGL}_2$ -linearized. On the other hand, consider the natural hyperplane bundle  $\mathcal{O}_{\mathbb{P}V_{1,4}}(1)$  on  $\mathbb{P}V_{1,4}$ . The element  $(-1, 1) \in \mathrm{SL}_2 \times \mathrm{PGL}_2$  acts on  $\mathcal{O}_{\mathbb{P}V_{1,4}}(1)$  by  $-1$ , so that the bundle  $\mathcal{H}' = \mathcal{H} \otimes \mathcal{O}_{\mathbb{P}V_{1,4}}(1)$  is  $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ -linearized. Then  $\mathbb{P}\mathcal{H}'$  is canonically isomorphic to  $\mathbb{P}\mathcal{H}$ . The

group  $\mathrm{PGL}_2 \times \mathrm{PGL}_2$  acts almost freely on  $\mathbb{P}V_{1,4}$ , for a general rational plane quartic has no nontrivial stabilizer in  $\mathrm{PGL}_3$  (cf. §3.4). Hence we may apply the no-name lemma to  $\mathcal{H}'$  to see that

(4.18)

$$\mathbb{P}\mathcal{H}/\mathrm{PGL}_2 \times \mathrm{PGL}_2 \sim \mathbb{P}\mathcal{H}'/\mathrm{PGL}_2 \times \mathrm{PGL}_2 \sim \mathbb{P}^8 \times (\mathbb{P}V_{1,4}/\mathrm{PGL}_2 \times \mathrm{PGL}_2).$$

In §3.4 we proved that  $\mathbb{P}V_{1,4}/\mathrm{PGL}_2 \times \mathrm{PGL}_2$  is rational. Therefore  $\mathbb{P}V_{1,8}/\mathrm{PGL}_2 \times \mathrm{PGL}_2$  is rational.

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